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COMMON FIXED POINT THEOREM OF AN INFINITE SEQUENCE OF MAPPINGS IN HILBERT SPACE

ARVIND KUMAR SHARMA[1], V. H. BADSHAH[2] & V. K. GUPTA[3]

[1]Department of Engineering Mathematics, M.I.T.M, Ujjain, Madhya Pradesh, India

[2]School of Studies in Mathematics, Vikram University, Ujjain, Madhya Pradesh, India

[3]Department of Mathematics, Government Madhav Science College, Ujjain, Madhya Pradesh, India

ABSTRACT

The aim of this present paper is to obtain a common fixed point for an infinite sequence of mappings on Hilbert space. Our purpose here is to generalize the our previous result [7]

KEYWORDS: Common Fixed Point, Hilbert Space, Infinite Sequence of Mappings

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1. INTRODUCTION

In 2011, Sharma Badshah and Gupta [7] have proved common fixed point theorem for the sequence $\{T_n\}_{n=1}^{\infty}$ of mappings satisfying the condition

$$\left\|T_i x - T_j y\right\| \le \alpha \frac{\left\|x - T_i x\right\|^2 + \left\|y - T_j y\right\|^2}{\left\|x - T_i x\right\| + \left\|y - T_j y\right\|} + \beta\left\|x - y\right\| \qquad (A)$$

$$\textit{for all } x, y \in S \textit{ and } x \ne y;\ \alpha \ge 0,\ \beta \ge 0 \textit{ and } 2\alpha + \beta < 1.$$

In 2005, Badshah and Meena [1] have proved common fixed point theorem for the sequence $\{T_n\}_{n=1}^{\infty}$ of mappings satisfying the condition

$$\left\|T_i x - T_j y\right\| \le \alpha \frac{\left\|x - T_i x\right\| . \left\|y - T_j y\right\|}{\left\|x - y\right\|} + \beta\left\|x - y\right\| \qquad (B)$$

$$\textit{for all } x, y \in S \textit{ with } x \ne y \textit{ also } \alpha \ge 0, \beta \ge 0 \textit{ and } \alpha + \beta < 1.$$

In 1991, Koparde and Waghmode [3] have proved common fixed point theorem for the sequence $\{T_n\}_{n=1}^{\infty}$ of mappings satisfying the condition

$$\left\|T_i x - T_j y\right\|^2 \le a\left(\left\|x - T_i x\right\|^2 + \left\|y - T_j y\right\|^2\right) \qquad (C)$$

$$\textit{for all } x, y \in S \textit{ and } x \ne y\ ; 0 \le a < \tfrac{1}{2}$$

Later in 1998, Pandhare and Waghmode [5] have proved common fixed point theorem for the sequence $\{T_n\}_{n=1}^{\infty}$ of mappings satisfying the condition

$$\left\|T_i x - T_j y\right\|^2 \le a\left\|x - T_i x\right\|^2 + b\left(\left\|x - T_i x\right\|^2 + \left\|y - T_j y\right\|^2\right) \tag{D}$$
$$for\ all\ x, y \in S\ and\ x \ne y; 0 \le a, 0 \le b < 1 and\ a + 2b < 1.$$

This result is generalizes by Veerapandi and Kumar [7] and the new condition is

$$\left\|T_i x - T_j y\right\|^2 \le a\left\|x - y\right\|^2 + b\left(\left\|x - T_i x\right\|^2 + \left\|y - T_j y\right\|^2\right) + \frac{c}{2}\left(\left\|x - T_i x\right\|^2 + \left\|y - T_j y\right\|^2\right) \tag{E}$$
$$for\ all\ x, y \in S\ and\ x \ne y\ where\ 0 \le a, b, c < 1\ and\ a + 2b + 2c < 1.$$

Now we introduce a new condition for the generalization of following known results.

Theorem 1: [7] Let S be a closed subset of a Hilbert space H and $\{T_n\}_{n=1}^{\infty} : S \to S$ be an infinite sequence of mappings satisfy (A). Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point.

Theorem 2: [1] Let S be a closed subset of a Hilbert space H and $\{T_n\}_{n=1}^{\infty} : S \to S$ be an infinite sequence of mappings satisfy (B). Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point.

Theorem 3: [3] Let S be a closed subset of a Hilbert space H and $\{T_n\}_{n=1}^{\infty} : S \to S$ be a sequence of mappings satisfy (C). Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point.

Theorem 4: [5] Let S be a closed subset of a Hilbert space H and $\{T_n\}_{n=1}^{\infty} : S \to S$ be a sequence of mappings satisfy (D). Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point.

Theorem 5: [8] Let S be a closed subset of a Hilbert space H and $\{T_n\}_{n=1}^{\infty} : S \to S$ be a sequence of mappings satisfy (E). Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point.

Main Result

We proved fixed point theorem for the infinite sequence $\{T_n\}_{n=1}^{\infty}$ to generalize our previous results [7].

Theorem: Let S be a closed subset of a Hilbert space H and $\{T_n\}_{n=1}^{\infty} : S \to S$ be an infinite sequence of mappings satisfying the following condition

$$\left\|T_i x - T_j y\right\| \le \left(\alpha + \beta \frac{\left\|x - T_i x\right\|}{\left\|x - y\right\|}\right)\left\|y - T_j y\right\| \tag{F}$$
$$for\ all\ x, y \in S\ and\ x \ne y;\ \alpha \ge 0,\ \beta \ge 0\ and\ 2\alpha + \beta < 1.$$

Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point.

Proof: Let S be a closed subset of a Hilbert space H and $\{T_n\}_{n=1}^{\infty} : S \to S$ be an infinite sequence of mappings. Let $x_0 \in S$ be any arbitrary point in S.

Define a sequence $\{x_n\}_{n=1}^{\infty}$ in S by

$$x_{n+1} = T_{n+1}x_n \,, \ for \ n = 0,1,2, ...$$

For any integer $n \geq 1$

$$\|x_{n+1} - x_n\| \ = \ \|T_{n+1}x_n - T_n x_{n-1}\|$$

$$\leq \left(\alpha + \beta \frac{\|x_n - T_{n+1}x_n\|}{\|x_n - x_{n-1}\|} \right) \|x_{n-1} - T_n x_{n-1}\|$$

$$\leq \left(\alpha + \beta \frac{\|x_n - x_{n+1}\|}{\|x_n - x_{n-1}\|} \right) \|x_{n-1} - x_n\|$$

$$\leq \alpha \|x_{n-1} - x_n\| + \beta \|x_n - x_{n+1}\|$$

$$i.e. \quad \|x_{n+1} - x_n\| \leq \alpha \|x_{n-1} - x_n\| + \beta \|x_n - x_{n+1}\|$$

$$\Rightarrow (1-\beta)\|x_{n+1} - x_n\| \leq \alpha \|x_n - x_{n-1}\|$$

$$\Rightarrow \|x_{n+1} - x_n\| \leq \frac{\alpha}{1-\beta} \|x_n - x_{n-1}\|$$

If $k = \dfrac{\alpha}{1\text{-}\beta}$ then $k < 1$.

$$\|x_{n+1} - x_n\| \leq k \|x_n - x_{n-1}\|$$

$$\leq k\|x_n - x_{n-1}\| \leq k^2 \|x_{n-1} - x_{n-2}\| \leq k^3 \|x_{n-2} - x_{n-3}\| \leq ... \leq k^n \|x_1 - x_0\|$$

$$i.e. \ \|x_{n+1} - x_n\| \leq k^n \|x_1 - x_0\| \quad for \ all \ \ n \geq 1 \ \ is \ \ integer.$$

Now for any positive integer $m \geq n \geq 1$

$$\|x_n - x_m\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + ... + \|x_{m-1} - x_m\|$$

$$\leq k^n \|x_1 - x_0\| + k^{n+1} \|x_1 - x_0\| + ... + k^{m-1} \|x_1 - x_0\|$$

$$\leq k^n \|x_1 - x_0\| \ \left(1 + k + ... + k^{m-n-1}\right)$$

$$i.e. \|x_n - x_m\| \leq \left(\frac{k^n}{1-k} \right) \|x_1 - x_0\| \ \ \to 0 \ as \, n \to \infty \ (k < 1)$$

Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Since S is a closed subset of a Hilbert space H, so $\{x_n\}_{n=1}^{\infty}$ converges to a point u in S.

Now we will show that u is common fixed point of infinite sequence $\{\mathrm{T}_n\}_{n=1}^{\infty}$ of mappings from S into S.

Suppose that $\mathrm{T}_n u \neq u$ for all *n*.

Consider for any positive integer *m* ($\neq n$)

$$\|u-\mathrm{T}_m u\| \leq \|u-x_n\|+\|x_n-\mathrm{T}_m u\|$$

$$=\|x_n-\mathrm{T}_m u\|$$

$$=\|\mathrm{T}_n x_{n-1}-\mathrm{T}_m u\|$$

$$\leq\left(\alpha+\beta\frac{\|x_{n-1}-\mathrm{T}_n x_{n-1}\|}{\|x_{n-1}-u\|}\right)\|u-\mathrm{T}_m u\|$$

$$\leq\left(\alpha+\beta\frac{\|x_{n-1}-x_n\|}{\|x_{n-1}-u\|}\right)\|u-\mathrm{T}_m u\|$$

$$\leq \ \alpha\|u-\mathrm{T}_m u\|+\beta\frac{\|x_{n-1}-x_n\|}{\|x_{n-1}-u\|}\|u-\mathrm{T}_m u\|$$

i.e. $$\|u-\mathrm{T}_m u\| \leq \ \alpha\|u-\mathrm{T}_m u\|+\beta\frac{\|x_{n-1}-x_n\|}{\|x_{n-1}-u\|}\|u-\mathrm{T}_m u\|$$

$$\Rightarrow \|u-\mathrm{T}_m u\| \leq \frac{\beta}{1-\alpha}\frac{\|x_{n-1}-x_n\|}{\|x_{n-1}-u\|}\|u-\mathrm{T}_m u\| \qquad \to 0 \ as \ n\to\infty$$

So $\|u-\mathrm{T}_m u\|\leq 0.$

Hence $u=\mathrm{T}_m u$ and so $u=\mathrm{T}_n u$ for all n.

Hence u is a common fixed point of infinite sequence $\{\mathrm{T}_n\}_{n=1}^{\infty}$ of mappings.

Uniqueness

.Suppose that there is $u \neq v$ such that $\mathrm{T_n} v = v$ for all n.

Consider $\| u-v \| = \| \mathrm{T_n} u-\mathrm{T_n} v\|$

$$\leq\left(\alpha+\beta\frac{\|u-\mathrm{T}_n u\|}{\|u-v\|}\right)\|v-\mathrm{T}_n v\|$$

i.e. $\| u-v \| \leq 0$

$\Rightarrow \quad \| u - v \| = 0$

Thus $\quad u = v$.

Hence fixed point is unique.

Example: Let X = [0, 1]**,** with Euclidean metric d .Then {X, d} is a Hilbert space with the norm defined by $d(x, y) = \|x - y\|$

Let $\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ be the sequence in X and let $\{\mathrm{T}_{\mathrm{n}}\}_{\mathrm{n}=1}^{\infty}$ be the infinite sequence of mappings such that

$x_{n+1} = \mathrm{T}_{n+1} x_n$, for $n = 0, 1, 2, \ldots$

Taking $x = \frac{1}{2^n}$ and $y = \frac{1}{2^{n-1}}$; $x \neq y$. Also $i = n+1$ and $j = n$.

Then from (F) $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in *X*, which is converges in *X* also it has a common point in *X*.

CONCLUSIONS

The theorem proved in this paper by using rational inequality is improved and stronger form of some earlier inequality given by Badshah and Meena [1], Sharma Badshah and Gupta [7], Koparde and Waghmode [3], Pandhare and Waghmode [5], Veerapandi and Kumar [8].

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